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Mode I Interaction of a Periodic Array of Parallel Cracks in a Functionally Graded Nonhomogeneous Plane

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The mode I interaction of a periodic array of parallel cracks which are uniformly spaced apart in a functionally graded material is investigated. The two-dimensional theory of non-homogeneous elasticity is employed as the basic framework for this study. The material nonhomogeneity is represented in terms of the spatial variation of the shear modulus in the exponential form along the direction of cracks, while Poisson's ratio is assumed to be constant. Formulation of the proposed mixed boundary value problem is reduced to solving a hypersingular integral equation with the crack surface displacement as a new unknown function. As a result, the variations of stress intensity factors are illustrated as a function of possible range of periodic crack spacing in conjunction with the different values of the material nonhomogeneity parameter. Furthermore, crack opening displaccements are presented for various geometric and material combinations.

Key Words: Periodic Array of Cracks, Functionally Graded Materials, Nonhomogeneous Properties, Hypersingular Integral Equation, Stress Intensity Factors

1. Introduction

As a new class of materials, functionally graded composites are evolving in which the microstructure is tailored in accordance with a predetermined composition profile to produce a continuous gradient or a gradual variation of properties with position (Koizumi, 1993). For instance, the graded thermomechanical properties obtained through the mixture of metals and ceramics exhibit material nonhomogeneities that can be effectively utilized to take advantage of both the high temperature characteristics of ceramics, and the fracture toughness capabilites of metals. The analysis of functionally graded materials can therefore be performed on a nonhomogeneous continuum basis.

Motivated by potential technological advances that can be achieved using this highly promising novel material system, especially in elevated thermal environments, considerable research effort is currently being directed toward resolving a variety of challenging issues in functionally graded, nonhomogeneous materials. This is because most existing theories and databases are for materials that possess uniform thermomechanical properties. In particular in the area of related fracture mechanics problems, Erdogan and his coworkers (e.g., Delale and Erdogan, 1983, 1988; Ozturk and Erdogan, 1993; Erdogan and Wu, 1993; Konda and Erdogan, 1994) have provided solutions to some basic boundary value problems entailing a crack in nonhomogeneous materials under mechanical loads, based on the premise that fatigue and fracture analysis and characterization of the new material system require the solutions to certain standard crack problems. Other previous investigations dealing with various forms of material nonhomogeneity are due to Dhaliwal and Singh (1978), Gerasoulis and strivastav (1980), Schovanec (1986), Ang and Clements (1987) and Eischen (1987). The aforementioned

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pioneering contributions have been later extended by Jin and Noda (1933) and Noda and Jin (1994) to include thermoelastic effects.

Although several issues related to certain crack problems for functionally graded, nonhomogeneous media have been resolved, the studies mentioned above are concerned with relatively simple cases of a single isolated crack or antiplane shear. The objective of this paper is to investigate the mode I problem of a periodic array of parallel cracks that are uniformly spaced apart in functionally graded media, using the theory of nonhomogeneous plane elasticity as our basis. Specifically, the shear modulus is assumed to vary exponentially in the direction of the cracks, and to simplify the analysis, Poisson's ratio is taken to be constant. Readers are referred to Benthem and Koiter (1973), Bowie (1973) and Nied (1987) for the solutions to similar problems of an infinite array of cracks in homogeneous media. By defining the crack surface siaplacement as an unknown auxiliary function, an integral equation with a strongly singular kernel is derived. Such a hypersingular integral equation is solved numerically by employing the concept of singular integrals interpreted in the finite-part sense (Hadamard, 1952). The stress intensity factors are defined from the stress fields that have square root singular behavior ahead of the crack tips and are evaluated in terms of the solutions to the integral equatiion. The main results presented are the variatios of stress intensity factors and crack opening displacements for different values of the nonhomogeneity parameter and the range of crack spacing.

2. Problem Statement and Formulation

Consider the system configuration illustrated schematically in Fig. 1, where an infinite array of cracks of identical length $2a$ are aligned parallel to the x-direction and equally spaced apart a distance $2h$ along the y-direction. For both convenience of analysis and practical considerations, the material nonhomogeneity of the functionally graded medium is represented such that the shear modulus μ and Poisson's ratio ν are approximat-

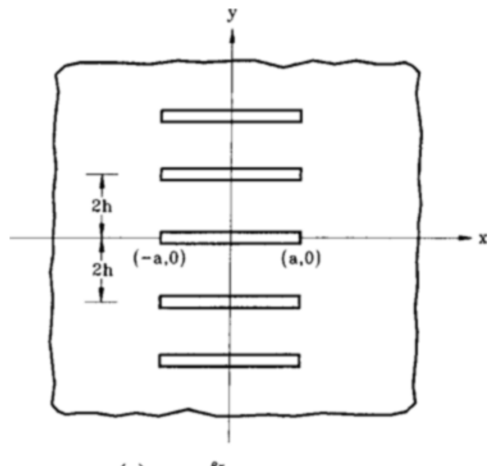

Fig. 1 Schematic representation of a periodic array of parallel cracks in a nonhomogeneous medium

ed as (Delale and Erdogan, 1983; Noda and Jin, 1994)

$$\mu(x)=\mu_o e^{\beta x}, \quad \nu=\text{constant} \tag{1}$$

where β is the nonhomogeneity parameter and μ_o is the shear modulus at $x=0$, i.e., $\mu_o=E_o/2(1+\nu)$ in which E_o corresponds to the constant elastic modulus.

With $u(x,y)$ and $v(x,y)$ denoting the displacement in the $x-$ and $y-$directions respectively, the governing equilibrium equations corresponding to the variable shear modulus in Eq. (1) are expressed as

$$\nabla^2 u+\frac{2}{\kappa-1}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y}\right) + \frac{\beta}{\kappa-1}\left[(1+\kappa)\frac{\partial u}{\partial x}+(3-\kappa)\frac{\partial v}{\partial y}\right]=0 \tag{2}$$

$$\nabla^2 v+\frac{2}{\kappa-1}\left(\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 v}{\partial y^2}\right) + \beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0 \tag{3}$$

where $\kappa=3-4\nu$ for the state of plane and $\kappa=(3-\nu)/(1+\nu)$ for the state of plane stress.

The general expressions of displacement components are readily obtained by as follows solving the above system of governing equations:

$$u(x,\ y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j m_j e^{n_j y-isx}ds \tag{4}$$

$$v(x, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j e^{n_j y-isx}ds \quad (5)$$

The stress components are obtained from the constitutive equations as

$$\sigma_{xx}(x, y)=\frac{\mu_0 e^{\beta x}}{2\pi(\varkappa-1)}\int_{-\infty}^{\infty}\sum_{j=1}^{4}[(3-\varkappa)n_j-i\cdot(1+\varkappa)sm_j]F_j e^{n_j y-isx}ds \quad (6)$$

$$\sigma_{yy}(x, y)=\frac{\mu_0 e^{\beta x}}{2\pi(\varkappa-1)}\int_{-\infty}^{\infty}\sum_{j=1}^{4}[(1+\varkappa)n_j-i\cdot(3-\varkappa)sm_j]F_j e^{n_j y-isx}ds \quad (7)$$

$$\tau_{xy}(x, y)=\frac{\mu_0 e^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}(m_j n_j-is)\cdot F_j e^{n_j y-isx}ds \quad (8)$$

where s is the Fourier transform variable, $F_j(s)$, $j=1, \ldots, 4$, are arbitrary unknowns, $i=(-1)^{1/2}$, and $n_j(s)$, $j=1, \ldots, 4$, are the roots of the characteristic equation:

$$[n^2-s(s+i\beta)]^2-\left(\frac{3-\varkappa}{1+\varkappa}\right)\beta^2 n^2=0 \quad (9)$$

It can be shown that

$$n_j=-\frac{1}{2}\left[(-1)^{j+1}\beta\sqrt{\frac{3-\varkappa}{1+\varkappa}}+\delta\cos\theta+i\delta\sin\theta\right]; \quad \mathrm{Re}(n_j)<0,\ j=1,2 \quad (10)$$

$$n_j=-\frac{1}{2}\left[(-1)^{j+1}\beta\sqrt{\frac{3-\varkappa}{1+\varkappa}}-\delta\cos\theta-i\delta\sin\theta\right]; \quad \mathrm{Re}(n_j)>0,\ j=3,4 \quad (11)$$

where $\theta(s)$ and $\delta(s)$ satisfy

$$\tan 2\theta=\frac{4\beta s(1+\varkappa)}{\beta^2(3-\varkappa)+4s^2(1+\varkappa)} \quad (12)$$

$$\delta=\left\{\left[\beta^2\left(\frac{3-\varkappa}{1+\varkappa}\right)+4s^2\right]^2+16\beta^2 s^2\right\}^{1/4} \quad (13)$$

and $m_j(s)$, $j=1, \ldots, 4$, are expressed as

$$m_j=\frac{(1+\varkappa)n_j^2+(1-\varkappa)(s+i\beta)s}{n_j[2is+\beta(1-\varkappa)]} \quad (14)$$

The medium in Fig. 1 is assumed to be loaded at $y=\pm\infty$ in the tensile mode. The locations $y=(2k+1)h$, $k=0, \pm 1, \pm 2, \ldots$, are then the planes of geometric and material symmetry along which the proper boundary conditions should satisfied, together with the self-equilibrated equivalent crack surface tractions. Consequently, for this mode I crack problem, it is sufficient to consider an infinte strip $|y|\leq h$ under the following set of boundary conditions:

$$v(x, h)=\tau_{xy}(x, h)=0;\ |x|<\infty \quad (15)$$

$$\tau_{xy}(x, 0)=0;\ |x|<\infty \quad (16)$$

$$\sigma_{yy}(x, 0)=\sigma(x);\ |x|<a \quad (17)$$

$$v(x, 0)=0;\ |x|>a \quad (18)$$

where $\sigma(x)$ describes the crack surface traction which acts as the only nonzero external load.

The four unknowns $F_j(s)$, $j=1,\ldots,4$, involved in the general solutions of elasticity equations can be obtained by applying the three homogeneous conditions in Eqs. (15) and (16) and the mixed conditions in Eqs. (17) and (18).

2.1 Singular integral equation

Solving the current crack problem entails the derivation of a certain integral equation. To this end, a new unknown function is defined as

$$\phi(x)=v(x, +0)-v(x, -0)=2v(x, 0);\ |x|<a \quad (19)$$

$$\phi(x)=0;\ |x|>a \quad (20)$$

and the applications of conditions in Eqs. (15), (16) and (19) allow the unknowns $F_j(s)$, $j=1, \ldots, 4$, to be obtained in terms of $\phi(x)$. Such an auxiliary function then becomes the only unknown that is to be determined from the remaining crack surface condition in Eq. (17).

After substituting the required unknowns as obtained in the above into Eq. (7), followed by some algebraic manipulations, the boundary condition in Eq. (17) can be written in the form of an integral equatiion:

$$\sigma_{yy}(x, 0)=\sigma(x)=\frac{\mu_0 e^{\beta x}}{2\pi(\varkappa-1)}\int_{-a}^{a}H(x, t)\phi(t)dt;\quad |x|<a \quad (21)$$

where $H(x, t)$ is the kernel function

$$H(x, t)=\int_{-\infty}^{\infty}\Lambda(s)e^{is(t-x)}ds \quad (22)$$

in which the integrand $\Lambda(s)$ is an intricate function of the elastic modulus and the geometry of the nonhomogeneous material as well as the Fourier transform variable s.

Subseqently, in order to investigate the singular nature of the integrl equation in (21), the asymptotic behavior of the integrand of the function $H(x, t)$ should be determined. Upon observing the behavior of $n_j(s)$ and $m_j(s)$, $j=1,...4$, for large values of $|s|$ as

$$\lim_{|s|\to\infty} n_j(s) = -\lim_{|s|\to\infty} n_{j+2}(s) = -|s|;\ j=1, 2 \quad (23)$$

$$\lim_{|s|\to\infty} m_j(s) = -\lim_{|s|\to\infty} m_{j+2}(s) = i\ \text{sgn}(s);\ j=1, 2 \quad (24)$$

where sgn(•) is the signum function, it can be further shown that the integrand $\Lambda(s)$ has the asymptotic property as $|s|$ tends to infinity such that

$$\lim_{|s|\to\infty} \Lambda(s) = \Lambda_\infty(s) \quad (25)$$

where the real and imaginary parts of the asymptotic value $\Lambda_\infty(s)$ are given as

$$\text{Re}\ \Lambda_\infty(s) = 2|s|\left(\frac{1-x}{1+x}\right) \quad (26)$$

$$\text{Im}\ \Lambda_\infty(s) = 0 \quad (27)$$

The foregoing asymptotic analysis makes it clear that any singularity the kernel $H(x, t)$ in Eq. (22) may have arises from the limiting behavior of $\Lambda(s)$ as $|s|$ approaches infinity. After separating the sigular part from the kernel and using the derivative of the Fourier integral representation of a generalized function (Friedman, 1969)

$$\frac{d}{dx}\int_{-\infty}^{\infty} \text{sgn}(s) e^{is(t-x)} ds = \frac{2i}{(t-x)^2} \quad (28)$$

the integral equation for $\phi(t)$ can be derived as

$$\int_{-a}^{a} \frac{\phi(t)}{(t-x)^2} dt + \int_{-a}^{a} p(x, t)\phi(t)dt = \frac{\pi(1+x)}{2\mu_o e^{\beta x}}\sigma(x);\ |x|<a \quad (29)$$

where $p(x, t)$ is a bounded kernel written as

$$p(x, t) = -\frac{1}{4}\left(\frac{1+x}{1-x}\right) \cdot \int_{-\infty}^{\infty} [\Lambda(s) - \Lambda_\infty(s)] e^{is(t-x)} ds \quad (30)$$

The first term in Eq. (29) is the integral with a strongly singular kernel $1/(t-x)^2$ that is to be interpreted as a finite-part integral in the sense of Hadamard (1952). With such dominant behavior of the kernel being regarded as hypersingular, it can be shown that the fundamental solution of the integral equation corresponds to the weight function of the Chebyshev polynomial of the second kind U_n (Kaya, 1984). In the normalized interval $\xi = x/a$ and $\eta = t/a$, the solution to the integral equation can therefore be expressed as

$$\phi(a\eta) = g(\eta)\sqrt{1-\eta^2}\ ;\ |\eta|<1 \quad (31)$$

where $g(\eta)$ is a bounded function which is nonzero at $\eta = \pm 1$, and can be approximated in terms of a truncated series of U_n such that

$$g(\eta) = \sum_{n=0}^{N} c_n U_n(\eta) \quad (32)$$

in which c_n, $0 \le n \le N$, are the unknown constants, which once evaluated can be directly used in conjunction with Eq. (19) to determine the shape of crack opening displacements.

Upon substituting Eq. (31) into Eq. (29) and using the following finite-part integral formula (Kaya, 1984)

$$\int_{-1}^{1} \frac{U_n(\eta)\sqrt{1-\eta^2}}{(\eta-\xi)^2} d\eta = -\pi(n+1)U_n(\xi);\quad n \ge 0,\ |\xi|<1 \quad (33)$$

the integral equation can be recast into functional form as

$$\sum_{n=0}^{N} c_n[-\pi(n+1)U_n(\xi) + h_n(\xi)] = \frac{\pi(1+x)a}{2\mu_o e^{\beta a\xi}}\sigma(a\xi);\ |\xi|<1 \quad (34)$$

where the function $h_n(\xi)$ is expressed as

$$h_n(\xi) = a^2\int_{-1}^{1} p(a\xi, a\eta) U_n(\eta)\sqrt{1-\eta^2} d\eta \quad (35)$$

A collocation technique is employed to solve the above functional equation for c_n, $0 \le n \le N$. Because of the nature of the problem and to ensure rapid convergence of results, the density of the collocation points near the singular points $\xi = \pm 1$ must be increased. An appropriate set of collocation points ξ_i which are concentrated near such end points is selected as

$$T_{N+1}(\xi_i) = 0,\ \xi_i = \cos\left(\frac{2i+1}{N+1}\frac{\pi}{2}\right);\quad i = 0, 1, 2, ..., N \quad (36)$$

where T_{N+1} is the Chebyshev polynomial of the first kind. A system of $N+1$ linearly independent equations to be solved for the $N+1$ unknown constants c_n , $0 \le n \le N$, is then obtained by evaluating the equaion in (34) at $N+1$ station points ξ_i.

2.2 Stress intensity factors

It was demonstrated in a rigorous manner by Delale and Erdogan (1983, 1988) that the singular behavior as well as the angular distribution of stresses around the crack tip in nonhomogeneous materials is identical to that in homogeneous materials, when the spatial distribution of elastic material properties is simply continuous near and at the crack tip. As a result, on the basis of this statement and the observation that the left-hand side of the integral equation in (29) provides the traction component $\sigma_{yy}(x, 0)$ on the entire x-axis, the mode I stress intensity factors $K_I(\pm a)$ at the crack tips $x = \pm a$ are defined as

$$K_I(+a) = \lim_{x \to a} \sqrt{2(x-a)}\, \sigma_{yy}(x, 0);\ x > a \quad (37)$$

$$K_I(-a) = \lim_{x \to -a} \sqrt{2(-x-a)}\, \sigma_{yy}(x, 0);\ x < -a \quad (38)$$

and, from Eqs. (29) and (31), can be evaluated in terms of $\phi(x)$ to

$$K_I(+a) = \frac{2\mu_o}{1+\kappa} \lim_{x \to a} \frac{e^{\beta x}}{\sqrt{2(a-x)}} \phi(x) = \frac{2\mu_o e^{\beta a}}{\sqrt{a}(1+\kappa)} \sum_{n=0}^{N} (n+1) c_n \quad (39)$$

$$K_I(-a) = \frac{2\mu_o}{1+\kappa} \lim_{x \to -a} \frac{e^{\beta x}}{\sqrt{2(a+x)}} \phi(x) = \frac{2\mu_o e^{-\beta a}}{\sqrt{a}(1+\kappa)} \sum_{n=0}^{N} (-1)^n (n+1) c_n \quad (40)$$

In addition, from Eqs. (19) and (31), the crack opening displacement can be obtained as

$$2v(x, 0) = \sqrt{1-\xi^2} \sum_{n=0}^{N} c_n U_n(\xi),\ \xi = \frac{x}{a} \quad (41)$$

3. Results and Discussion

Numerical results are obtained for the two types of loading that involve the prescribed strain and traction. Fixed-grip strain loading is then assumed in the absence of cracks as $\varepsilon_{yy}(x, \pm\infty) = \varepsilon_o + (x/a)\varepsilon_1$ where the first constant and the second linear terms, respectively, correspond to the uniform strain and the bending applied at $y = \pm\infty$. Hence, via a proper superposition, the equivalent crack surface traction in Eqs. (17) and (29) can be exressed as

$$\sigma_{yy}(x, 0) = \sigma(x)$$

$$= -\frac{8\mu_o e^{\beta x}}{1+\kappa}\left[\varepsilon_o + \left(\frac{x}{a}\right)\varepsilon_1\right];\ |x| < a \quad (42)$$

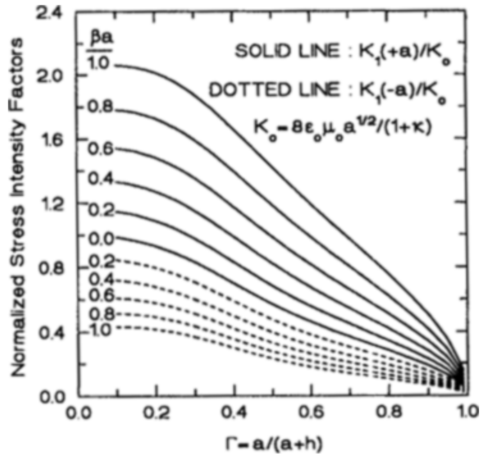

Fig. 2 Variation of normalized stress intensity factors $K_I(\pm a)/K_o$ with the crack spacing $\Gamma = a/(a+h)$ for different values of the nonhomogeneity parameter βa under uniform strain ε_o: $\sigma(x) = -8\varepsilon_o\mu_o e^{\beta x}/(1+\kappa)$ and $\nu = 0.3$

and for the sake of comparison, a uniform traction σ_o applied on the crack surface is also assumed:

$$\sigma_{yy}(x, 0) = \sigma(x) = -\sigma_o;\ |x| < a \quad (43)$$

For the loading conditions as described above, the resulting values of normalized stress intensity factors $K_I(\pm a)/K_o$ are illustrated in Figs. 2~4 as a function of the geometric ratio $\Gamma = a/(a+h)$ for different values of the dimensionless nonhomogeneity parameter βa. The normalizing factors K_o for each of the loadings are specified as follows:

$K_o = 8\varepsilon_o\mu_o\sqrt{a}/(1+\kappa)$ for the uniform strain loading ε_o (44)

$K_o = 8\varepsilon_1\mu_o\sqrt{a}/(1+\kappa)$ for the linear strain loading $(x/a)\varepsilon_1$ (45)

$K_o = \sigma_o\sqrt{a}$ for the uniform traction loading σ_o (46)

which are the stress intensity factors for the case of a single isolated crack ($\Gamma = 0.0$) in an infinite homogeneous plane ($\beta a = 0.0$).

A state of plane strain is assumed with Poisson's ratio set to $\nu=0.3$. In the examples considered, as many as forty terms in Eqs. (34) and (36) are required to ensure accurate solutions, due to the rather slow convergence for large values of βa. The integralsin Eqs. (30) and (35) are evaluated by employing the Gauss-Legendre and Gauss-Chebyshev quadratures, respectively (Davis and Rabinowitz, 1984). One remark at this point is that the difference between the plane strain and plane stress solutions is negligible, being at most less than one percent when the degree of material nonhomogeneity and the crack spacing are relatively large, e.g., $\beta a=1.0$ and $\Gamma=0.1$.

To confirm the soundness and validity of our analytical procedure and numerical results, the values of stress intensity factors are checked with the special cases available in the literature. Specifically, the present results for a homogeneous material with $\beta a=0.0$ are in excellent agreement with those compiled by Murakami (1987) and/or Rooke and Cartwright (1976), while the values obtained for nonhomogeneous materials with $\beta a \neq 0.0$ are similar to those given by Dalale and Erdogan (1983) when the ratio Γ approaches zero, as demonstrated in Tables 1 and 2.

The stress intensity factors obtained for the uniform strain ε_o are plotted in Fig. 2. To be observed from this figure is that the crack tips at $x=+a$ which are on the stiffer side of the material are, in general subjected to greater stress intensification as an increasing function of the nonhomogeneity parameter βa over the given range of Γ. On the other side of the crack tips at $x=-a$, however, the severity of such stress intensity factors is alleviated as βa increases. Such an effect of βa is shown to be more pronounced when the cracks are spaced further apart. In particular, the decrease in the crack spacing, in conjunction with βa, results in the variation of crack interactions such that the magnitudes of stress intensity factors are substantially reduced as Γ approaches unity, to a level well below those for a single crack of the same length. This trend is most noteworthy for the crack tips $x=+a$ when $\beta a=1.0$.

Under the sole application of a linear strain component $(x/a)\varepsilon_1$, Fig. 3 predicts the negative stress intensity factors at $x=-a$ due to the antisymmetry of the applied loading, implying compressive singular stresses and the closure of cracks. Even though the implicit assumption of frictionless and open cracks may therefore be invalidated, such negative values can be useful when the superposition due to a sufficiently large

Table 1 Normalized stress intensity factors $K_I(\pm a)/K_o$ in a homogeneous material ($\beta a=0.0$) for different values of the crack spacing $\Gamma=a/(a+h)$ under uniform loadings: $\sigma(x)=-8\varepsilon_o\mu_o/(1+\varkappa)$ for uniform strain ε_o or $\sigma(x)=-\sigma_o$ for uniform traction σ_o and $\nu=0.3$

$\Gamma=0.1$		$\Gamma=0.2$		$\Gamma=0.3$		$\Gamma=0.4$		$\Gamma=0.5$	
Present	Ref.(+)	Present	Ref.	Present	Ref.	Present	Ref.	Present	Ref.
0.9851	0.9851	0.9309	0.9318	0.8311	0.8315	0.7005	0.7005	0.5702	0.5703

(+): Murakami (1987)

Table 2 Normalized stress intensity factors $K_I(\pm a)/K_o$ for a periodic array of parallel cracks ($\Gamma>0.0$) and a single isolated crack ($\Gamma>0.0$) for the nonhomogeneity parameter $\beta a=1.0$ under uniform loadings: $\sigma(x)=-8\varepsilon_o\mu_o e^{\beta x}/(1+\varkappa)$ for uniform strain ε_o, $\sigma(x)=-\sigma_o$ for uniform traction σ_o and $\nu=0.3$

Types of loading	$\Gamma=0.3$		$\Gamma=0.2$		$\Gamma=0.1$		$\Gamma=0.0$(*)	
	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$
Uniform strain ε_o	1.8781	0.3686	2.0144	0.4158	2.0601	0.4319	2.063	0.433
Uniform traction σ_o	1.0314	0.6742	1.1636	0.7231	1.2088	0.7395	1.222	0.745

(*): Delale and Erdogan (1983)

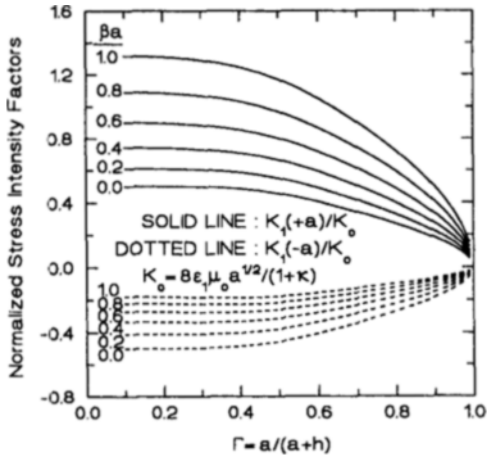

Fig. 3 Variation of normalized stress intensity factors $K_I(\pm a)/K_o$ with the crack spacing $\Gamma = a/(a+h)$ for different values of the nonhomogeneity parameter βa under linear strain $(x/a)\varepsilon_1$: $\sigma(x) = -8\varepsilon_{1x}\mu_o e^{\beta x}/a(1+x)$ and $\nu = 0.3$

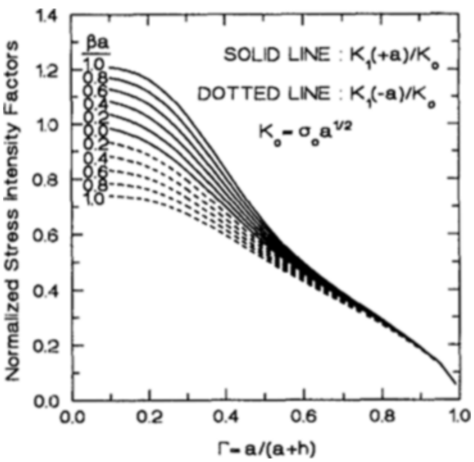

Fig. 4 Variation of normalized stress intensity factors $K_I(\pm a)/K_o$ with the crack spacing $\Gamma = a/(a+h)$ for different values of the nonhomogeneity parameter βa under uniform traction σ_o: $\sigma(x) = -\sigma_o$ and $\nu = 0.3$

tensile loading gives rise to positive resultants.

Figure 4 shows the variations of stress intensity factors obtained under a uniform crack surface traction σ_o. Similar observations can be made with regard to the influences of geometric and material variables, Γ and βa, as was done in Fig. 2. By comparison, however, the normalized stress intensity factors of these uniformly pressurized

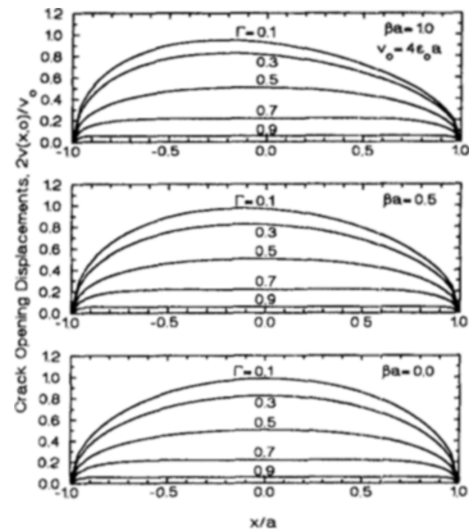

Fig. 5 Normalized crack opening displacements $2v(x, 0)/v_o$ for different values of the crack spacing $\Gamma = a/(a+h)$ and the nonhomogeneity parameter βa under uniform strain ε_o: $\sigma(x) = -8\varepsilon_o\mu_o e^{\beta x}/(1+x)$ and $\nu = 0.3$

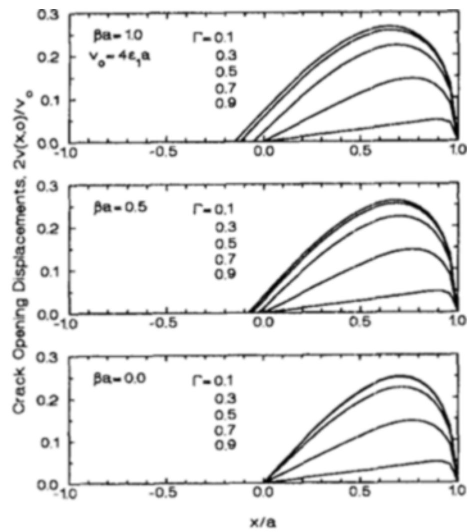

Fig. 6 Normalized crack opening displacements $2v(x, 0)/v_o$ for different values of the crack spacing $\Gamma = a/(a+h)$ and the nonhomogeneity parameter βa under linear strain $(x/a)\varepsilon_1$: $\sigma(x) = -8\varepsilon_{1x}\mu_o e^{\beta x}/a(1+x)$ and $\nu = 0.3$

cracks are seen to be affected to a lesser extent by the nonhomogeneity parameter βa.

In Figs. 5~7, crack opening displacements are provided for the uniform strain ε_o, the linear

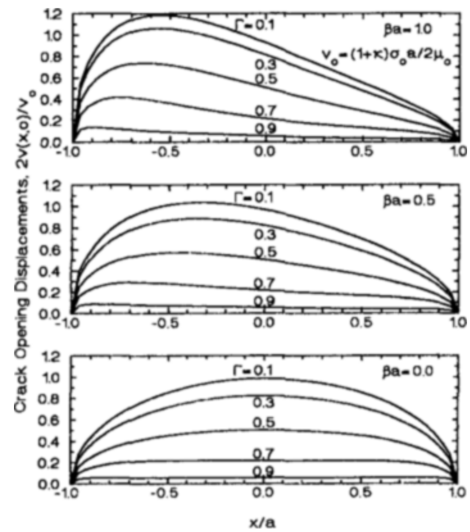

Fig. 7 Normalized crack opening displacements $2v(x, 0)/v_o$ for different values of the crack spacing $\Gamma=a/(a+h)$ and the nonhomogeneity parameter βa under uniform traction σ_o: $\sigma(x)=-\sigma_o$ and $\nu=0.3$

Table 3 Effect of Poisson's ratio ν on the normalized stress intensity factors $K_I(\pm a)/K_o$ for different values of the crack spacing $\Gamma=a/(a+h)$ and the nonhomogeneity parameter βa under uniform strain ε_o: $\sigma(x)=-8\varepsilon_o\mu_o e^{\beta x}/(1+\chi)$

$\Gamma=0.2$						
ν	$\beta a=0.0$		$\beta a=0.5$		$\beta a=1.0$	
	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$
0.05	0.9309	0.9309	1.3678	0.6245	1.9875	0.4077
0.10	0.9309	0.9309	1.3686	0.6249	1.9921	0.4090
0.20	0.9309	0.9309	1.3704	0.6259	2.0023	0.4121
0.30	0.9309	0.9309	1.3726	0.6271	2.0144	0.4158
0.40	0.9309	0.9309	1.3755	0.6287	2.0292	0.4202
0.45	0.9309	0.9309	1.3773	0.6297	2.0379	0.4227

$\Gamma=0.5$						
ν	$\beta a=0.0$		$\beta a=0.5$		$\beta a=1.0$	
	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$
0.05	0.5702	0.5702	0.9008	0.3617	1.4244	0.2303
0.10	0.5702	0.5702	0.9008	0.3617	1.4245	0.2302
0.20	0.5702	0.5702	0.9008	0.3617	1.4247	0.2302
0.30	0.5702	0.5702	0.9008	0.3617	1.4249	0.2301
0.40	0.5702	0.5702	0.9008	0.3617	1.4252	0.2300
0.45	0.5702	0.5702	0.9008	0.3617	1.4253	0.2300

Table 4 Effect of Poisson's ratio ν on the normalized stress intensity factors $K_I(\pm a)/K_o$ for different values of the crack spacing $\Gamma=a/(a+h)$ and the nonhomogeneity parameter βa under uniform strain σ_o: $\sigma(x)=-\sigma_o$

$\Gamma=0.2$						
ν	$\beta a=0.0$		$\beta a=0.5$		$\beta a=1.0$	
	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$
0.05	0.9309	0.9309	1.0441	0.8190	1.1407	0.7135
0.10	0.9309	0.9309	1.0448	0.8195	1.1446	0.7151
0.20	0.9309	0.9309	1.0465	0.8206	1.1533	0.7188
0.30	0.9309	0.9309	1.0485	0.8219	1.1636	0.7231
0.40	0.9309	0.9309	1.0512	0.8236	1.1760	0.7283
0.45	0.9309	0.9309	1.0528	0.8247	1.1832	0.7315

$\Gamma=0.5$						
ν	$\beta a=0.0$		$\beta a=0.5$		$\beta a=1.0$	
	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$	$K_I(+a)/K_o$	$K_I(-a)/K_o$
0.05	0.5702	0.5702	0.6042	0.5398	0.6430	0.5120
0.10	0.5702	0.5702	0.6042	0.5398	0.6429	0.5120
0.20	0.5702	0.5702	0.6042	0.5398	0.6428	0.5121
0.30	0.5702	0.5702	0.6041	0.5398	0.6426	0.5122
0.40	0.5702	0.5702	0.6041	0.5398	0.6424	0.5122
0.45	0.5702	0.5702	0.6041	0.5398	0.6422	0.5123

strain $(x/a)\varepsilon_1$, and the uniform crack surface traction σ_o, respectively. To be noted from these figures is that the decrease in the spacing among nearby interacting cracks greatly restrains the crack opening, leading to an increasingly flat slope along the axes of cracks. Figures 5 and 7 further demonstrate that as βa increases, the displacements are being shifted from the shape of symmetric curves in the homogeneous medium to that of skewed curves with respect to the centerline of the cracks, where the less stiff portion of the nonhomogenous medium has larger crack opening displacements than the homogeneous medium. Such an asymmetric crack shape is particularly noticeable for the nonhomogeneous medium that is subjected to uniform crack surface traction as illustrated in Fig. 7. With the imposition of a linear strain, it is shown in Fig. 6 that the left halves of the cracks are closed, consistent with the negative values of stress intensity factors at $x=-a$ as given in Fig. 3.

Some additional results are presented Tables 3 and 4 to examine the effect of variations of Poisson's ratio ν on the stess intensity factors for different values of Γ and βa. Each table is for the medium under a uniform strain ε_o and uniform crack surface traction σ_o. It is observed that the effect of ν appears to become more notable for greater values of βa and smaller values of Γ. The difference in magnitudes of the stress intensity factors for the given range of $0.05 \leq \nu \leq 0.45$ may be estimated to be as high as 3.6 percent when $\beta a = 1.0$ and $\Gamma = 0.2$. It is likely, however, that the values of ν vary within an even narrower range than the above so that the effect of Poisson's ratio is negligibly small. As a result, similar to the previous findings for a single crack in a nonhomogeneous medium (Delale and Erdogan, 1983; Ozturk and Erdogan, 1993), the assumption of neglecting the possible spatial variation of Poisson's ratio in the nonhomogeneous medium is not a very restrictive one that can be physically justified.

4. Closure

The analysis of functionally graded materials containing a periodic array of parallel cracks has been performed within the framework of nonhomogeneous plane elasticity. The nonhomogeneous property distribution was approximated by expressing the elastic modulus in the form of an exponential function that renders the current crack problem amenable to analytical treatment via the use of Fourier transform techniques. In consequence, a hypersingular integral equation was derived with the crack surface displacement as a new unknown function. The main emphasis was then placed upon the evaluation of stress intensity factors from the square root singular behavior at the crack tips, based on standard methods of linear elastic fracture mechanics. Some selected numerical values were illustrated under fixed-grip strain and uniform crack surface loadings. As a result, the parametric effects of crack spacing were addressed in conjunction with the material nonhomogeneity and different values of Poisson's ratio. The crack opening displacements were also presented for various geometric and material combinations to provide further physical into the behavior of an array of multiple cracks in functionally graded, nonhomogeneous media.

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